

A class of non-ergodic probabilistic cellular automata with unique invariant measure and quasi-periodic orbit

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Abstract

We provide an example of a discrete-time Markov process on the three-dimensional infinite integer lattice with \mathbb{Z}_q -invariant Bernoulli-increments which has as local state space the cyclic group \mathbb{Z}_q . We show that the system has a unique invariant measure, but remarkably possesses an invariant set of measures on which the dynamics is conjugate to an irrational rotation on the continuous sphere S^1 . The update mechanism we construct is exponentially well localized on the lattice.

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1 Introduction

The possible long-time behavior of infinite lattice systems under stochastic dynamics is a subject of ongoing research. The situation is non-trivial already for reversible dynamics and becomes even more difficult when one leaves the assumption of reversibility of the dynamics and enters the realm of driven systems. Infinite lattice systems may possess different equilibria for the same stochastic dynamics. The first question which comes to mind is to estimate the approach to equilibrium, if there is a unique equilibrium (using for example spectral gap analysis, logarithmic Sobolev inequalities, etc. see [7, 8]). If there are multiple equilibria one may be interested in their domains of attraction. An interesting question in this context is whether a unique equilibrium has to be an attractor for a stochastic lattice dynamics (see for example [42, 6, 44, 46]). Ideally one would like to understand possible behavior of invariant sets and attractors. Under what circumstances can there be oscillatory behavior and closed orbits of measures? This is difficult to answer for infinite lattice systems.

On the one hand motivation comes from the study of given models describing e.g systems of coupled neurons. These show characteristic patterns of spiking, phases of long-range order, periodicity, synchronization as well as disordered phases. A mathematical analysis of these interesting phenomena so far has been restricted mainly to mean-field models (see for example [11] or the Kuramoto model [10, 1, 28, 29]). On the other hand there is the theoretical interest to make progress by restricting the possible forms of limiting behavior, or to explore possible forms and provide model systems which illustrate possible forms of "non-standard" limit behavior. Ultimately one may strive for a classification of models into classes which look similar, but in the beginning we are forced to work with model systems, which serve as illustrations, their universality being a question that hopefully can be tackled later.

These types of question may be posed for continuous-time Markov processes or discrete-time Markov processes. In a previous paper [31] we were able to settle an old question raised by Liggett [42] whether it is possible to have a continuous-time Markovian dynamics with a unique invariant measure which does not attract all initial conditions, but has a closed orbit of measures. An example of a probabilistic cellular automaton (PCA) which shows that non-ergodicity is possible in one dimension with positive rates was given in [26]. An example of a PCA which shows that non-ergodicity in one dimension, even with a unique stationary measure, is possible was given in [6] (albeit with some deterministic updatings). In [9] another example of a non-ergodic, non-degenerate, Ising-type PCA on the d -dimensional lattice with $d \geq 2$ is exhibited. Both examples have periodic orbits of period two. To compare, in the case of an interacting particle system (IPS) where the updating is in continuous time, non-ergodicity with unique invariant measure was proved to be impossible for local rates in one lattice dimension in [44, 46]. In two lattice dimensions the question remains open. For general background, see also [41, 49].

The aim of the present paper is to provide an example of a discrete-time Markov process with *discrete* \mathbb{Z}_q -invariant Bernoulli-updates having a unique invariant measure, but a non-trivial quasi-periodic orbit under the dynamics. It is remarkable that a quasi-periodic orbit which is conjugate to an irrational rotation on a sphere can arise from exponentially localized dynamical rules on a discrete local spin space.

To do so, we consider a class of discrete q -state spin equilibrium models defined in terms of a translation-invariant quasilocal specification with discrete clock-rotation invariance having extremal Gibbs measures μ'_φ labeled by the uncountably many values of φ in the one-dimensional sphere (introduced in [15]). Next we construct an associated discrete-time Markov process as a PCA with exponentially localized updating rule. The process has the property to reproduce a deterministic rotation of the extremal Gibbs measures and preserve macroscopic coherence by a quasilocal and time-synchronous updating mechanism in discrete time without deterministic transitions. We prove that, depending on an updating-velocity parameter τ there is either a continuum of time-stationary measures and closed orbits of rotating states, or a unique time-stationary measure and a dense orbit of rotating states. In both cases the process is non-ergodic.

Our paper partially builds on constructions of [31] where we considered an IPS which can be seen as a continuous-time analogue of the present Markov process. The bulk of the present work consists in the proof of uniqueness of the invariant measure in the absence of continuous time, which heavily draws on techniques relating entropy loss, the inverse transition operator and the Gibbs property of the invariant measure.

1.1 Discrete-time infinite lattice models – Main result

We consider Markovian dynamics on the infinite-volume state space $\Omega = E^G$, where E is a finite local state space sitting at each site $i \in G$ and G is a countable set. In the examples we want to construct we will specifically choose $E = \mathbb{Z}_q = \{0, 1, \dots, q-1\}$ to be the cyclic group and $G = \mathbb{Z}^d$ to be the d -dimensional integer lattice, so that the infinite volume configuration space Ω is an abelian group w.r.t. sitewise addition modulo q .

Recall the following definitions. A (*deterministic*) *cellular automaton* is given by a deterministic local updating rule $f : E^\Lambda \mapsto E$ where $\Lambda \subset G$ is a finite set containing the origin $\Lambda \ni 0$. The simultaneous application of f yields a corresponding function $F : \Omega \mapsto \Omega$ acting on infinite-volume configurations $\sigma \in \Omega$, where the i -th coordinate of the image $F(\sigma)$ is given by $F(\sigma)_i = f(\theta_i \sigma)$ for all sites $i \in G$ and $\theta_i \sigma$ is the configuration obtained from σ by a lattice shift by the vector $i \in G$, that is $(\theta_i \sigma)_j = \sigma_{j+i}$. One then likes to study properties of the discrete dynamical system obtained by iterates of F . We will not discuss such deterministic cellular automata any further, but mention that there are recent developments concerning very interesting questions about simple cellular automata which are fed random initial configurations, bootstrap percolation being one of these (see for example [2, 5, 3, 12]).

A (*strict*) *probabilistic cellular automaton* (PCA) is given by a probabilistic single-site updating kernel p from the infinite-volume state space Ω to single-site probability measures, which plays the role of a random version of the map f . So one has $p : \Omega \mapsto \mathcal{P}(E)$ where $\mathcal{P}(E)$ denotes the probability measures on the finite set E and p is assumed to be a strictly local probability kernel, which means that for each $a \in E$ the functions $\sigma \mapsto p(\sigma, a)$ are strictly local functions for all $a \in E$, that is functions depending only on finitely many coordinates of σ . This probabilistic updating rule is applied (stochastically) independently over all sites to yield a Markovian transition operator M on infinite lattice configurations (which generalizes the deterministic function F) of the form

$$M(\sigma, d\eta) = \otimes_{i \in G} p(\theta_i \sigma, d\eta_i).$$

This transition operator describes simultaneous updates on the infinite lattice which are performed in a sitewise independent way according to a lattice-shift invariant rule. There is a huge literature on probabilistic cellular automata (see for example [6, 41, 26, 9, 49, 48]).

A *probabilistic cellular automaton with exponentially localized update kernel* is given by a Markovian kernel M from the set of infinite-volume configurations Ω to the probability measures on Ω allowing for exponentially suppressed non-localities in the following sense.

1. The update kernel $M(\sigma, d\eta) =: M_\sigma(d\eta)$ is *uniformly spatially mixing in the future with exponent α_2* . This means by definition that there is a positive constant K_2 , such that at any fixed starting configuration σ the following decay estimate holds: For all finite volumes $\Lambda \subset \Delta \subset G$, for all sets A in the sigma-algebra generated by the η -coordinates in Λ and sets B in the sigma-algebra generated by the η -coordinates in $G \setminus \Delta$ one has

$$|M_\sigma(A|B) - M_\sigma(A)| \leq K_2 \sum_{i \in \Lambda, j \in G \setminus \Delta} e^{-\alpha_2 |i-j|}. \quad (1)$$

We remark that this uniform mixing property is much stronger (closer to independence) than simply correlation decay. While correlation decay holds e.g whenever the conditional measure on the η variables is an extremal Gibbs measure even in the phase-transition regime, phase transitions are excluded by (1).

2. The update kernel M satisfies the property of *exponential locality from the past with exponent α_1* . This means by definition that there is a positive constant K_1 such that the variation w.r.t. σ_i of the probability $M(\sigma, \eta_\Lambda)$ with finite Λ is bounded by the exponential estimate

$$\delta_i(M(\cdot, \eta_\Lambda)) := \sup_{\substack{\sigma, \tilde{\sigma}: \\ \sigma_{i^c} = \tilde{\sigma}_{i^c}}} (M(\sigma, \eta_\Lambda) - M(\tilde{\sigma}, \eta_\Lambda)) \leq K_1 e^{-\alpha_1 |i-\Lambda|}.$$

Here we used notation $i^c = G \setminus \{i\}$, $|i - \Lambda| = \min_{j \in \Lambda} |i - j|$ and $\eta_\Lambda := \{\xi \in \Omega : 1_{\eta_\Lambda}(\xi) = 1\}$.

Notice that kernels with these two properties are related to the infinite-volume Markovian kernels studied in [40].

Specifying now to Markov kernels M on the state space $(\mathbb{Z}_q)^G$ we say that M is \mathbb{Z}_q -invariant iff it is compatible with joint rotation on the local state space and we can write $M(\sigma + a, A + a) = M(\sigma, A)$ for all $a \in \mathbb{Z}_q$. We say that M has *Bernoulli-increments* if $M(\sigma, \sigma + \{0, 1\}^G) = 1$. In that case the updated configuration is obtained from an initial configuration σ by the site-wise addition modulo q of a $\{0, 1\}^G$ -valued field N of Bernoulli-increments whose distribution conditional on σ we denote by the symbol with a hat, namely $\hat{M}(\sigma, dn)$. To exclude degeneracies we only consider kernels in this paper which are *uniformly non-null* meaning by definition that $\hat{M}_\sigma(n_\Lambda) \geq c^{|\Lambda|}$ for all $n_\Lambda \in \{0, 1\}^\Lambda$ for some strictly positive uniform constant c .

Given such a Bernoulli updating kernel $\hat{M}(\sigma, dn)$ from $(\mathbb{Z}_q)^{\mathbb{Z}^3}$ to $\{0, 1\}^{\mathbb{Z}^3}$, we then look at the associated discrete-time Markov process

$$X^t = (X^{t-1} + N^t) \bmod q$$

on the state space $\Omega = (\mathbb{Z}_q)^{\mathbb{Z}^3}$ with increment distribution

$$\mathcal{L}(N^t = \cdot | X^{t-1} = \sigma) = \hat{M}(\sigma, \cdot)$$

and use the symbol M (without the hat) for the corresponding transition kernel, i.e.

$$M(\sigma, \cdot) = \mathcal{L}(X^t = \cdot | X^{t-1} = \sigma).$$

We write $\mathcal{P}_\theta(\Omega)$ for the lattice-translation invariant probability measures on Ω .

The main result of the paper is the following theorem which states that we may construct exponentially localized transition kernels with Bernoulli updates which are arbitrarily well localized, but whose time-evolutions show macroscopic coherence and non-ergodicity in the PCA-sense.

Theorem 1.1 *For any arbitrarily large prescribed mixing exponents $\alpha_1, \alpha_2 \in (0, \infty)$ there exists an integer $q_0 \in \mathbb{N}$ such that, for all $q \geq q_0$ the following is true.*

There exists an updating kernel M with Bernoulli increments on $(\mathbb{Z}_q)^{\mathbb{Z}^3}$, which satisfies the properties of uniform exponential mixing in the future with an exponent of (at least) α_2 and uniform spatial locality from the past with an exponent of (at least) α_1 , such that the associated discrete-time stochastic dynamics possesses a set $\mathcal{I} \subset \mathcal{P}_\theta(\Omega)$ of lattice translation-invariant measures on which the dynamics acts like a rotation. More precisely:

1. *There is a quasilocal observable $\psi : \Omega \rightarrow \mathbb{R}^2$ such that the expectation map*

$$\nu \mapsto \nu(\psi)$$

is a bijection from \mathcal{I} to the sphere $S^1 \subset \mathbb{R}^2$.

2. The stochastic dynamics restricted to the invariant set \mathcal{I} is conjugate to a rotation by some angle $\tau = \tau(M)$:

$$\forall \nu \in \mathcal{I} : \quad (M\nu)(\psi) = \nu(\psi) + \tau$$

where we have written rotation on S^1 on the r.h.s in an additive way.

3. The kernel M can always be chosen so that the cases, $\tau/2\pi$ is rational, or irrational can both occur.

- (a) If $\tau/2\pi$ is irrational, there exists a unique invariant measure ν^* among the translation-invariant measures, but the dynamics is quasiperiodic. In particular $M^n \nu \not\rightarrow \nu^*$ (in the sense of local convergence) and the PCA is non-ergodic.
- (b) If $\tau/2\pi$ is rational, there are uncountably many periodic orbits and uncountably many invariant measures.

1.2 Comparison with continuous-time dynamics

We believe that it is instructive to compare this result to our previous work on the existence of non-ergodic *continuous-time* dynamics in [31]. For easy comparison let us also present the following result which is based on the work of [31].

We consider a continuous-time interacting particle system (IPS) on $(\mathbb{Z}_q)^{\mathbb{Z}^3}$ given in terms of a generator U acting on observables $\psi : (\mathbb{Z}_q)^{\mathbb{Z}^3} \mapsto \mathbb{R}$ of the form

$$U\psi(\omega) = \sum_{i \in \mathbb{Z}^d} \left(c_i^+(\omega) (\psi(\omega + 1_i) - \psi(\omega)) + c_i^-(\omega) (\psi(\omega - 1_i) - \psi(\omega)) \right) \quad (2)$$

where $(\omega \pm 1_i)_j = (\omega_j \pm \delta_{i,j}) \bmod q$, so that it allows both for increments ± 1 at each site i .

In analogy to the notion of uniform exponential locality from the past in the discrete-time setup we say that the rates $c_i^\pm(\omega)$ satisfy the property of *exponential locality with exponent α_1* whenever there is a finite K_1 such that

$$\delta_i c_j^\pm(\cdot) := \sup_{\substack{\omega, \tilde{\omega}: \\ \omega_{ic} = \tilde{\omega}_{ic}}} (c_j^\pm(\omega) - c_j^\pm(\tilde{\omega})) \leq K_1 e^{-\alpha_1 |i-j|}. \quad (3)$$

This in particular makes the dynamics well-defined by standard methods. Then we have the following theorem.

Theorem 1.2 *For arbitrarily large exponent $\alpha_1 \in (0, \infty)$ there exists an integer $q_0 \in \mathbb{N}$ such that, for all $q \geq q_0$ the following is true: There exists a generator Q on $\Omega = (\mathbb{Z}_q)^{\mathbb{Z}^3}$ with lattice-translation invariant and \mathbb{Z}_q -invariant rates which satisfy exponential locality with exponent (at least) α_1 such that the associated Markov process with continuous-time semigroup $(S_t)_{t \geq 0}$ possesses a set $\mathcal{I} \subset \mathcal{P}_\theta(\Omega)$ of lattice translation-invariant measures on which the dynamics acts like a rotation. More precisely:*

1. *(This is identical to the corresponding point for the discrete-time dynamics.)
There is a quasilocal observable $\psi : \Omega \rightarrow \mathbb{R}^2$ such that the expectation map*

$$\nu \mapsto \nu(\psi)$$

is a measurable bijection from \mathcal{I} to the sphere $S^1 \subset \mathbb{R}^2$.

2. *The continuous-time stochastic dynamics restricted to the invariant set \mathcal{I} is conjugate to a continuous rotation by $t \in \mathbb{R}$:*

$$\forall \nu \in \mathcal{I} : \quad (S_t \nu)(\psi) = \nu(\psi) + t.$$

3. *There is a unique time-stationary measure ν^* among the lattice-translation invariant measures, namely the uniform mixture over \mathcal{I} .*

Note that in the case of the continuous-time Markov process we do not need a requirement analogous to property (1) posed in the PCA case.

Neither does Theorem 1.1 imply Theorem 1.2 nor does Theorem 1.2 imply Theorem 1.1.

1.3 Ideas of the proof

There are common parts (see A below) and essential differences (see C, and the more difficult D) in the treatment of discrete-time dynamics and continuous-time dynamics.

A. First we give a one-parameter family of measures $\mathcal{I} = \mathcal{I}(\beta, q) \subset \mathcal{P}_\theta((\mathbb{Z}_q)^{\mathbb{Z}^3})$, depending on an inverse temperature parameter β for each sufficiently large q . This family is used in both cases of discrete-time and continuous-time dynamics. We will shortly review its construction which was given in [31], based on arguments concerning the preservation of Gibbsianness, in Section 2.1.

B. We construct a discrete-time Bernoulli update kernel for which \mathcal{I} is an invariant set in Section 2.2. This is analogous to but different from the construction of $U = U(\beta, q)$ given in [31].

C. We prove locality, mixing, and further properties also in Section 2.2.

D. We prove uniqueness of the invariant measure (where it is claimed to hold). The discrete case does not follow from the continuous case since time-derivatives are not available, and it necessitates the use of a different chain of arguments.

Physically the construction of the rotating-states mechanism is inspired by conjectures in [43] in the context of IPS based on a clock model in an intermediate-temperature regime [25]. To carry out our construction, as in [31, 32] we draw on the relation to the planar rotator model which has as a local state space the one-dimension sphere S^1 . On the lattice in three or more space dimensions, at sufficiently strong coupling constant this system exhibits the breaking of the rotation symmetry in spin-space, see [24, 45, 23, 43]. To arrive at a system of discrete spins (or particles) with finite local state space \mathbb{Z}_q a local discretization is applied for q sufficiently large but finite. Then the interplay between the systems

of discrete and continuous spins is exploited. In particular we use the fact that the discretization map bijectively maps the lattice translation-invariant extremal Gibbs measures μ_φ of the continuous system to the extremal lattice translation-invariant Gibbs measures μ'_φ of the discrete system where φ runs over the one-dimensional sphere S^1 . Note also the non-trivial fact that the discrete system has uncountably many extremal Gibbs measures.

As we will see below we can define an associated discrete-time Markov process with transition kernel M_τ , where τ is a continuous parameter carrying also the meaning of an angle. The kernel M_τ assigns to a particle configuration in $(\mathbb{Z}_q)^{\mathbb{Z}^d}$ a random particle configuration in $(\mathbb{Z}_q)^{\mathbb{Z}^d}$ and will be obtained by a natural three-step procedure. We call this procedure *Sample-Rotate-Project*, to be described in detail below. Here it is the rotation step which carries the dependence on the continuous angle τ . For $0 \leq \tau \leq 2\pi/q$ the updating is Bernoulli. The dynamics works nicely on the Gibbs measures and we have the rotation property:

1. An application of the transition operator M_τ to a discrete Gibbs measure μ'_φ yields a rotation by an angle τ , so that we have $M_\tau \mu'_\varphi = \mu'_{\varphi+\tau}$.

The proof of this fact is more straightforward than the proof of the analogous statement in the IPS setup [31], given the previous work on preservation of Gibbsianness under discretizations. Property 1 already implies that the symmetric mixture $\mu'_* = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \mu'_\varphi$ is invariant under the dynamics.

Note also that we can play with the velocity-parameter τ now, and consider the action of the dynamics on the Gibbs measures. Rational values of $\tau/2\pi$ yield finite closed orbits, of which there are uncountably many, so that there are uncountably many time-stationary measures obtained as the equal-weight measures on these orbits. Irrational values of $\tau/2\pi$ yield a quasiperiodic orbit and a unique time-stationary measure. Next we note:

2. The translation-invariant measures μ' which have no decrease of relative entropy density relative to the invariant measure μ'_* (equivalently, to one of the Gibbs measures μ'_φ) in one time step, are necessarily Gibbs measures for the same specification as μ'_* .

In particular translation-invariant and dynamically stationary measures are Gibbs measures for the same potential as μ'_* and their relative entropy density is also zero. In short: Zero entropic loss implies Gibbsianness. To use such a connection is similar in spirit to the IPS case (where the analogous connection was termed "Holley's argument" [30, 42]). However, the proof is different and in fact we get stronger statements than for the IPS case (compare Theorem 3.3 below). We are inspired in this part by the paper [9] about probabilistic cellular automata with strictly local update rules (see also [40]). We also use the relation between, on the one hand, the decrease of relative entropy density between a general starting measure and the invariant measure μ'_* under application of the dynamics, and on the other hand, relative entropies between corresponding time-reversed transition operators. Using Künsch's ideas from [40] we then derive the desired DLR equation to identify the Gibbs measures. Technically there are also differences

in our treatment to these papers: We need to take proper care of non-localities, but we are able to bypass a Künsch-type representation of the transition operator in terms of double-Gibbs potentials and work directly with specifications, taking advantage of their properties we have to our disposition in our case, which considerably simplifies things. (For background on how to go from specifications to potentials see [47, 34, 35].)

The non-reversible time-evolutions we consider here suggest another set of questions, namely whether there are any non-Gibbsian pathologies along the trajectories depending on starting measures as found for reversible dynamics in [16, 18, 39, 36, 20, 17, 14, 33, 22].

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2 Equilibrium model and rotation dynamics

In this section we present the equilibrium model also exhibited in [31]. Further we define the updating via the three-step procedure *Sample-Rotate-Project* and show locality properties.

2.1 The equilibrium model

The first-layer model: We have to first introduce a continuous-spin model which is given in terms of a Gibbsian specification for an absolutely summable Hamiltonian acting on lattice-configurations with continuous local state space. More precisely we consider an S^1 -rotation invariant and translation-invariant Gibbsian specification γ^Φ on the lattice $G = \mathbb{Z}^d$, with local state space $S^1 = [0, 2\pi)$. Let this specification $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda \subset G}$ be given in the standard way by an absolutely summable, S^1 -invariant and translation-invariant potential $\Phi = (\Phi_A)_{A \subset G, A \text{ finite}}$, w.r.t to the Lebesgue measure λ on the spheres. This means that the Gibbsian specification is given by the family of probability kernels

$$\gamma_\Lambda^\Phi(B|\eta) = \frac{\int 1_B(\sigma_\Lambda \eta_{\Lambda^c}) \exp(-H_\Lambda(\sigma_\Lambda \eta_{\Lambda^c})) \lambda^{\otimes \Lambda}(d\sigma_\Lambda)}{\int \exp(-H_\Lambda(\sigma_\Lambda \eta_{\Lambda^c})) \lambda^{\otimes \Lambda}(d\sigma_\Lambda)}$$

for finite $\Lambda \subset G$ and Hamiltonian $H_\Lambda = \sum_{A \cap \Lambda \neq \emptyset} \Phi_A$ applied to a measurable set $B \subset (S^1)^G$ and a boundary condition $\eta \in (S^1)^G$ (for details on Gibbsian specifications see [27]). We use notation $\Lambda^c := G \setminus \Lambda$. A standard example of such a model is provided by the nearest-neighbor scalarproduct interaction rotator model with Hamiltonian

$$H_\Lambda(\sigma_\Lambda \eta_{\Lambda^c}) = -\beta \sum_{i,j \in \Lambda: i \sim j} \cos(\sigma_i - \sigma_j) - \beta \sum_{i \in \Lambda, j \in \Lambda^c: i \sim j} \cos(\sigma_i - \eta_j). \quad (4)$$

Denote by $\mathcal{G}(\gamma^\Phi)$ the simplex of the Gibbs measures corresponding to this specification, which are the probability measures μ on $(S^1)^G$ which satisfy the DLR-equation $\int \mu(d\eta) \gamma_\Lambda^\Phi(B|\eta) = \mu(B)$ for all finite Λ . Denote by $\mathcal{G}_\theta(\gamma^\Phi)$ the lattice translation-invariant Gibbs measures.

We will make as an assumption on the class of potentials (Hamiltonians) we discuss moreover that it has a continuous symmetry breaking in the following sense. Assume that the extremal translation-invariant Gibbs measures can be obtained as weak limits with homogeneous boundary conditions, i.e with $\eta_\varphi \in (S^1)^G$ defined as $(\eta_\varphi)_i = \varphi$ for all $i \in G$ and $\varphi \in S^1$ we have

$$\text{ex } \mathcal{G}_\theta(\gamma^\Phi) = \{\mu_\varphi | \mu_\varphi = \lim_{\Lambda \nearrow G} \gamma_\Lambda^\Phi(\cdot | \eta_\varphi), \varphi \in S^1\}.$$

We further assume that different boundary conditions η_φ yield different measures so that there is a unique labelling of states μ_φ by the angles φ in the sphere S^1 . It is a non-trivial proven fact that this assumption is true in the case of the standard rotator model (4) in $d = 3$ for λ -a.a temperatures in the low-temperature region as discussed in [24, 43, 45]. Here the unique labelling can be given by the local magnetization $\mu_\varphi(\sigma_0) = m e_\varphi$ where $1 > m > 0$ is the temperature-dependent length of the unit vector $e_\varphi \in S^1$ with angle φ .

The second-layer model: We will now describe the discretization transformation which maps the continuous-spin model to a discrete-spin model. Denote by T the local coarse-graining with equal arcs, i.e $T : [0, 2\pi) \mapsto \{1, \dots, q\}$ where $T(\varphi) := k$ iff $2\pi(k-1)/q \leq \varphi < 2\pi k/q$. Extend this map to infinite-volume configurations by performing it sitewise. We will refer to the image space $\Omega := \{0, \dots, q-1\}^G$ as the coarse-grained layer. In particular we will consider images of infinite-volume measures under T .

We will need to choose the parameter of this discretization $q \geq q_0(\Phi)$ large enough so that the image measures of first-layer Gibbs measures are again Gibbs measures for a discrete specification on the coarse-grained layer. That this is always possible follows from the earlier works [38, 15]. More precisely, we assume that the condition from Theorem 2.1 of [15] is fulfilled (ensuring a regime where the Dobrushin uniqueness condition holds for the so-called constrained first-layer models where the Dobrushin condition is a weak dependence condition implying uniqueness and locality properties). Note, as in our notation the usual inverse temperature parameter β is incorporated into Φ , for β tending to infinity so does $q_0(\Phi)$.

To talk about the correspondence between the continuous and the discrete system we need to make explicit the relevant Gibbsian specification for the latter. To do so define a family of kernels $\gamma' = (\gamma'_\Lambda)_{\Lambda \subset G, \Lambda \text{ finite}}$ for the discretized model by

$$\begin{aligned} \gamma'_\Lambda(\sigma'_\Lambda | \sigma'_{\Lambda^c}) &= \frac{\int \mu_{\Lambda^c}[\sigma'_{\Lambda^c}](d\sigma_{\Lambda^c}) \int \lambda^{\otimes \Lambda}(d\sigma_\Lambda) e^{-H_\Lambda(\sigma_\Lambda \sigma_{\Lambda^c})} 1_{T(\sigma_\Lambda) = \sigma'_\Lambda}}{\int \mu_{\Lambda^c}[\sigma'_{\Lambda^c}](d\sigma_{\Lambda^c}) \int \lambda^{\otimes \Lambda}(d\sigma_\Lambda) e^{-H_\Lambda(\sigma_\Lambda \sigma_{\Lambda^c})}} \\ &= \frac{\mu_{\Lambda^c}[\sigma'_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\sigma'_\Lambda}))}{\mu_{\Lambda^c}[\sigma'_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda}))} \end{aligned} \tag{5}$$

where the second line is a short notation that we want to adapt in the sequel. The measure $\mu_{\Lambda^c}[\sigma'_{\Lambda^c}]$ is the unique continuous-spin Gibbs measure for a system on the smaller volume Λ^c with conditional specification obtained by deleting all interactions with Λ and constrained to take values σ_{Λ^c} with discretization images $T(\sigma_{\Lambda^c}) = \sigma'_{\Lambda^c}$. For more details and definition of $\mu_{\Lambda^c}[\sigma'_{\Lambda^c}]$ in terms of formulas see [31] Section 2. Note that these constrained Gibbs measures are well-defined and have nice locality properties for sufficiently fine discretization $q \geq q_0(\Phi)$, see [15, 38] and below. For general background on constrained Gibbs measures in the context of preservation of Gibbsianness see [13, 21, 37]. We note that γ' is indeed a quasilocal specification and the discretized Gibbs measures are Gibbs for γ' .

The relation between first- and second-layer models: The infinite-volume discretization map T is injective when applied to the set of translation-invariant extremal Gibbs states in the continuum model $\text{ex } \mathcal{G}_\theta(\gamma^\Phi)$. More precisely we have the following proposition proved in [31].

Proposition 2.1 *Let $q \geq q_0(\Phi)$, then T is a bijection from $\text{ex } \mathcal{G}_\theta(\gamma^\Phi)$ to $\text{ex } \mathcal{G}_\theta(\gamma')$ with inverse given by the kernel $\mu_G[\sigma'](d\sigma)$.*

Here $\mu_G[\sigma'](d\sigma)$ is the unique conditional continuous-spin Gibbs measure on the whole volume G . Regarding part 1 of Theorem 1.1 we have the following corollary.

Corollary 2.2 *Let Φ be in the phase-transition region, $q \geq q_0(\Phi)$ and $m = |\mu(\sigma_0)|$ the uniform local magnetization length for all $\mu \in \text{ex } \mathcal{G}_\theta(\gamma^\Phi)$. Under these assumptions, the mapping $\psi : \Omega \mapsto S^1$*

$$\psi(\sigma') := \mu_G[\sigma'](\sigma_0)/m$$

is quasilocal and the expectation map $\nu \mapsto \nu(\psi)$ is a measurable bijection from $\text{ex } \mathcal{G}_\theta(\gamma')$ to the sphere $S^1 \subset \mathbb{R}^2$.

Proof: The quasilocality of ψ follows from Dobrushin uniqueness arguments given in [31] for $q \geq q_0(\Phi)$. By the continuous symmetry breaking of the first layer model and the surjectivity of T , for every $\varphi \in S^1$ there exists a $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$. Further we have

$$\mu'_\varphi(\psi) = \frac{1}{m} \int_{\Omega} \mu'_\varphi(d\sigma') \mu_G[\sigma'](\sigma_0) = \frac{1}{m} \mu_\varphi(\sigma_0) = e_\varphi$$

and different φ yield different μ'_φ . □

2.2 The updating mechanism

We define an updating at finite time $0 \leq \tau < 2\pi$, according to the *Sample-Rotate-Project* algorithm, as follows: Given a discrete-spin configuration σ' we perform the following steps:

1. Sample a continuous-spin configuration σ according to the conditional measure $\mu_G[\sigma'](d\sigma)$.
2. Rotate deterministically the resulting continuous-spin configuration $\sigma \mapsto \sigma + \tau$ jointly in all sites by the same angle τ .
3. Project the rotated configuration using the discretization map T , i.e look at the coarse-grained configuration $T(\sigma + \tau)$.

The resulting kernel on discrete spins, describing the probability distribution of $T(\sigma + \tau)$ where σ is distributed according to $\mu_G[\sigma'](d\sigma)$ for a given initial configuration σ' , we denote by $M_\tau(\sigma', \cdot)$. This *transition operator* can be expressed via

$$M_\tau(\sigma', \eta'_\Lambda) = \mu_G[\sigma'](\eta'_{\Lambda, \tau}) \quad (6)$$

where Λ is a finite set of sites and

$$\eta'_{\Lambda, \tau} := T^{-1}(\eta'_\Lambda) - \tau \in (S^1)^\Lambda.$$

In words: $\eta'_{\Lambda, \tau}$ is obtained by joint rotation by $-\tau$ of the segments of the sphere prescribed by η'_Λ . It will be convenient to use the rewriting

$$M_\tau(\sigma', \eta'_\Lambda) = \frac{\mu_{\Lambda^c}[\sigma'_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\sigma'_\Lambda} 1_{\eta'_{\Lambda, \tau}}))}{\mu_{\Lambda^c}[\sigma'_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\sigma'_\Lambda}))} \quad (7)$$

which follows from reorganization of terms in the Hamiltonian in the Dobrushin uniqueness regime, see [31] Section 2.

Notice that $M_\tau(\sigma', \eta'_\Lambda) = 0$ if and only if $\sigma'_{\Lambda, 0} \cap \eta'_{\Lambda, \tau} = \emptyset$. If $M_\tau(\sigma', \eta'_\Lambda) > 0$ we call the configuration η' *accessible for σ' in Λ* . For $0 \leq \tau \leq 2\pi/q$ the updating has indeed Bernoulli-increments since in this case the numerator of (7) is zero for $\eta'_\Lambda \neq \sigma'_\Lambda + \{0, 1\}^\Lambda$.

The following two propositions verify the locality properties addressed in Theorem 1.1.

Proposition 2.3 *For any $\alpha_2 > 0$ we can choose the discretization $q = q(\alpha_2)$ fine enough such that the update kernel $M_\tau(\sigma', d\eta')$ is uniformly spatially mixing in the future with exponent $\alpha_2 > 0$.*

Proof: Let σ' be any starting configuration. Further let A be a set of configurations measurable w.r.t the sigma-algebra corresponding to a finite volume Λ and B be a set of configurations measurable w.r.t the sigma-algebra corresponding to a volume Δ^c where Δ is a finite volume. Further assume $M_\tau(\sigma', B) > 0$, then we have

$$M_\tau(\sigma', A|B) - M_\tau(\sigma', A) = \mu_G[\sigma'](B_\tau)^{-1} \mu_G[\sigma'](A_\tau \cap B_\tau) - \mu_G[\sigma'](A_\tau).$$

where we used notation analogue to (6) for A_τ, B_τ . Notice, $\mu_G[\sigma']$ is uniquely specified by the constrained specification $\gamma^{\sigma'}$ (for details see [31]) and thus we can write

$$\mu_G[\sigma'](A_\tau \cap B_\tau) = \int_{B_\tau} \gamma_\Delta^{\sigma'}(A_\tau|\sigma) \mu_G[\sigma'](d\sigma).$$

$\gamma^{\sigma'}$ is in the Dobrushin uniqueness region with Dobrushin matrix \bar{C} uniformly in σ' thus by Theorem 8.23 (ii) in [27] we have

$$\sup_{A_\tau} \|\gamma_\Delta^{\sigma'}(A_\tau|\cdot) - \mu_G[\sigma'](A_\tau)\| \leq \sum_{i \in \Lambda, j \in \Delta^c} \bar{D}_{ij}$$

with $\bar{D} := \sum_{n \geq 0} \bar{C}^n$. Now we can pick $q \geq q(\alpha_2)$ large enough such that the Dobrushin matrix has exponential decay (see [27] Remark 8.26 and [31] Lemma 2.8) and hence $\bar{D}_{ij} \leq K_2 e^{-\alpha_2|i-j|}$ which finishes the proof. \square

Proposition 2.4 *The update kernel M_τ satisfies the property of exponential locality from the past with exponent α_1 .*

Proof: Using notation as in (6) we have

$$\delta_i(M(\cdot, \eta'_\Lambda)) = \sup_{\substack{\sigma', \tilde{\sigma}': \\ \sigma'_{i^c} = \tilde{\sigma}'_{i^c}}} |\mu_G[\sigma'](\eta'_{\Lambda, \tau}) - \mu_G[\tilde{\sigma}'](\eta'_{\Lambda, \tau})| \leq \sum_{j \in \Lambda} \bar{D}_{ji}$$

since $\mu_G[\sigma']$ is the unique Gibbs measure for the specification $\gamma^{\sigma'}$ which is in the Dobrushin uniqueness region see Theorem 8.20 in [27]. As above we can choose $q \geq q(2\alpha_1)$ to be large enough such that $\bar{D}_{ji} \leq C_1 e^{-2\alpha_1|i-j|}$ and thus for $i \notin \Lambda$ and α_1 sufficiently large

$$\begin{aligned} \sum_{j \in \Lambda} \bar{D}_{ji} &\leq C_1 \sum_{j \in \Lambda} e^{-2\alpha_1|i-j|} \leq C_1 \sum_{k=|i-\Lambda|}^{\infty} e^{-2\alpha_1 k} \#\{j \in \mathbb{Z}^d, |j-i| = k\} \\ &\leq C_2 e^{-\alpha_1|i-\Lambda|}. \end{aligned}$$

\square

As already addressed in Section 1.2, in [31] we present a rotation dynamics in continuous-time as an IPS. Specifically we exhibit a Markov generator L with the property that for the associated semigroup $(S_t)_{t \geq 0}$ we have $S_t \mu'_\varphi = \mu'_{\varphi+t}$ where $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$ (see [31] Theorem 1.3). The exponential locality property (3) is a consequence of Lemma 3.4 in [31]. More precisely, given any arbitrarily large decay exponent α_1 we may ensure the decay (3) by choosing β and q in [31] such that the corresponding Dobrushin-constant $\bar{c} = \bar{c}(\beta, q)$ (of formula (28) of [31]) is sufficiently small. Now choose β sufficiently large such that the initial rotator models has a phase transition. Since $\bar{c}(\beta, q) \downarrow 0$ for β fixed, with $q \uparrow \infty$, the statement now follows for q large.

In the present discrete-time setting the mapping $\tau \mapsto M_\tau$ can not be expected to be a semigroup. In particular at finite τ the transition operator M_τ differs from the continuous rotation semigroup $(S_t)_{t \geq 0}$. They become equal at infinitesimal τ and this is the route for the definition of $(S_t)_{t \geq 0}$ (see the proof of Theorem 1.3 in [31] Section 3.2). Nevertheless M_τ also possesses the rotation property presented in the following proposition.

Proposition 2.5 *Discrete Gibbs measures transform in a covariant way, i.e $M_\tau \mu'_\varphi = \mu'_{\varphi+\tau}$ for all $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$ and $0 \leq \tau < 2\pi$.*

In particular $\mathcal{I} := \mathcal{G}_\theta(\gamma')$ is the set of translation-invariant measures where the discrete-time process acts like a rotation. Applying the function ψ from Section 2.1 this proves part 1 and 2 in Theorem 1.1.

Proof: It suffices to prove that for all $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$ and discrete-spin test-functions f the following equality holds:

$$\int \mu'_\varphi(d\sigma') \int M_\tau(\sigma', d\eta') f(\eta') = \int \mu'_{\varphi+\tau}(d\eta') f(\eta')$$

By Proposition 2.1 the conditional probability under coarse-graining of a continuous-spin measure ordering in direction φ does not depend on φ , i.e for continuous-spin test-functions g we have

$$\int \mu_\varphi(d\sigma) g(\sigma) = \int \mu'_\varphi(d\sigma') \mu[\sigma'](d\sigma) g(\sigma). \quad (8)$$

Thus we can write

$$\begin{aligned} \int \mu'_\varphi(d\sigma') \int M_\tau(\sigma', d\eta') f(\eta') &= \int \mu'_\varphi(d\sigma') \int \mu[\sigma'](d\sigma) f(T(\sigma + \tau)) \\ &= \int \mu_\varphi(d\sigma) f(T(\sigma + \tau)) \\ &= \int \mu_{\varphi+\tau}(d\sigma) f(T(\sigma)) \\ &= \int \mu'_{\varphi+\tau}(d\eta') f(\eta') \end{aligned}$$

where the first equality is the definition of M_τ , the second equality is the independence of conditional probability on φ (see (8)), the third equality is the property of the continuous-spin Gibbs measures to transform under rotations and the last inequality is the definition of the coarse-grained measures. \square

To summarize the interplay between the discretization and the dynamics let us consider the joint rotation of a first-layer measure μ by an angle τ written as $R_\tau \mu$. Then, under R_τ and the Markov transition operator $M_\tau(\sigma', d\eta')$, the diagram in Figure 1 is commutative.

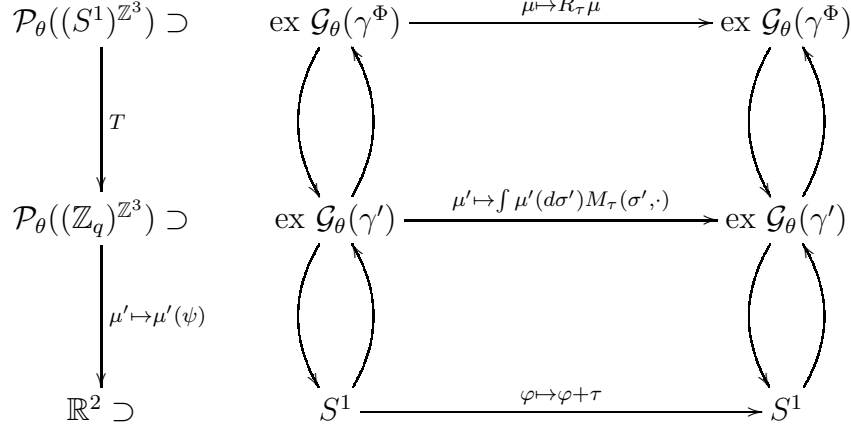


Figure 1: The discretization map T and labelling map $\mu' \mapsto \mu'(\psi)$ become bijections when applied to the extremal translation-invariant Gibbs measures. The transition kernel M_τ reproduces the deterministic rotation actions R_τ and $\varphi \mapsto \varphi + \tau$.

In order to further explain the transition kernel M_τ , let us consider the case of the zero initial continuous-spin Hamiltonian as an example. In this case the spins become independent and it is sufficient to consider a single spin with the Gibbs distribution being the uniform distribution on the circle $\sigma_0 \sim \text{Unif}([0, 2\pi])$. The image measure under T of the continuous-spin Gibbs measure is again the uniform distribution on the discretized circle $\sigma'_0 \sim \text{Unif}(\{1, \dots, q\})$. Hence, for $0 \leq \tau < 2\pi/q$ we have that

$$M_\tau(\sigma'_0, \{\sigma'_0, \sigma'_0 + 1\}) = 1 \quad \text{and} \quad M_\tau(\sigma'_0, \sigma'_0 + 1) = \tau \frac{q}{2\pi}.$$

This describes a random walk $(\sigma'_0(n\tau))_{n \in \mathbb{N}}$ on the discretized circle $\{1, \dots, q\}$ which can only move up by one step with probability proportional to τ or stay where it is. Clearly for larger parameters τ it is more probable to jump. In that sense τ controls the velocity of the walker.

Taking the limit $\tau \downarrow 0$ the process converges to a Poisson process which moves around the discretized circle where jumps are made with intensity $q/2\pi$.

Finally let us point out that the uniformly mixed Gibbs measure $\mu'_* := \frac{1}{2\pi} \int_0^{2\pi} d\varphi \mu'_\varphi$ with $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$ is time-stationary for M_τ . Indeed we have

$$M_\tau \mu'_* = \frac{1}{2\pi} \int_0^{2\pi} d\varphi M_\tau \mu'_\varphi = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \mu'_{\varphi+\tau} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \mu'_\varphi = \mu'_*. \quad (9)$$

In other words, there exists a translation-invariant Gibbs measure that is also time-stationary for the dynamics. In the following section we prove that any translation-invariant and time-stationary measure must be a Gibbs measure for the same specification.

3 Entropic loss and reversed transition operators

We want to connect the two properties, having zero entropy loss under time evolution and being Gibbs w.r.t the same specification as μ'_* . In order to do this relative entropy densities will be considered. More precisely we want to employ an extension to our Gibbsian updating mechanism of arguments which are carried out in [9] Proposition 2.1 and Proposition 2.2 involving time-reversed transition operators. From now on we will always consider translation-invariant measures.

Let us introduce some notation. For infinite-volume probability measures $\nu', \mu' \in \mathcal{P}(\{0, \dots, q-1\}^G)$ and a finite set of sites Λ the *local relative entropy* is defined as

$$h_\Lambda(\nu'|\mu') := \sum_{\sigma'_\Lambda \in \{1, \dots, q\}^\Lambda} \nu'(1_{\sigma'_\Lambda}) \log \frac{\nu'(1_{\sigma'_\Lambda})}{\mu'(1_{\sigma'_\Lambda})}.$$

If $\mu' \in \mathcal{G}_\theta(\gamma')$, existence of the *specific relative entropy*

$$h(\nu'|\mu') := \limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} h_\Lambda(\nu'|\mu')$$

is guaranteed, where Λ varies over hypercubes centered at the origin. Similarly one can define the *specific relative entropy between transition operators w.r.t a base measure ν'* by

$$\mathcal{H}_{\nu'}(M|\tilde{M}) = \int \nu'(d\sigma') \limsup_{\Lambda} \frac{1}{|\Lambda|} h_\Lambda(M(\sigma', \cdot) | \tilde{M}(\sigma', \cdot)).$$

Let us define the *joint two-step distribution* $Q_{\nu'}(d\sigma', d\eta') := M_\tau(\sigma', d\eta')\nu'(d\sigma')$. We will consider different conditionings of $Q_{\nu'}$. To keep notation reasonably simple we set the convention and write σ' for the *present configuration* and η' for the *future configuration*, just as in $Q_{\nu'}(d\sigma', d\eta')$. With this we have $Q_{\nu'}(\eta'_\Lambda | \sigma') = M_\tau(\sigma', \eta'_\Lambda)$ which is independent of ν' . The *backwards transition operator* is given by

$$\hat{M}_{\tau, \nu'}(\eta', \sigma'_\Lambda) := Q_{\nu'}(\sigma'_\Lambda | \eta').$$

Using the short notation $M_\tau \nu'(\eta'_\Lambda) = \int \nu'(d\sigma') M_\tau(\sigma', \eta'_\Lambda)$, the backwards transition operator is characterized by the requirement that

$$\int \nu'(d\sigma') \int M_\tau(\sigma', d\eta') f(\sigma', \eta') = \int M_\tau \nu'(d\eta') \int \hat{M}_{\tau, \nu'}(\eta', d\sigma') f(\sigma', \eta') \quad (10)$$

holds for all local test-functions f .

Lemma 3.1 *The backwards transition operator for any translation-invariant Gibbs measure is given by $M_{-\tau}$ where $M_{-\tau}$ is obtained from formula (6) for negative τ .*

Proof: Let us first check (10) for the extremal Gibbs measure, i.e let $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$, then independently of the angle φ we have

$$\begin{aligned}
\int \mu'_\varphi(d\sigma') \int M_\tau(\sigma', d\eta') f(\sigma', \eta') &= \int \mu'_\varphi(d\sigma') \int \mu_G[\sigma'](d\eta) f(\sigma', T(\eta + \tau)) \\
&= \int \mu'_\varphi(d\sigma') \int \mu_G[\sigma'](d\eta) f(T(\eta), T(\eta + \tau)) \\
&= \int \mu_\varphi(d\eta) f(T(\eta), T(\eta + \tau)) \\
&= \int \mu_{\varphi+\tau}(d\eta) f(T(\eta - \tau), T(\eta)) \\
&= \int \mu'_{\varphi+\tau}(d\eta') \int \mu_G[\eta'](d\sigma) f(T(\sigma - \tau), \eta') \\
&= \int \mu'_{\varphi+\tau}(d\eta') \int M_{-\tau}(\eta', d\sigma') f(\sigma', \eta')
\end{aligned}$$

where we used equation (8) two times. By linearity of the integrals the above equation also holds for any convex combination of the extremal Gibbs measures and hence for all $\mu' \in \mathcal{G}_\theta(\gamma')$. \square

Next we consider the entropy loss under M_τ which can be expressed in terms of the backward transition operators.

Lemma 3.2 *Suppose that μ' is a translation-invariant Gibbs measure w.r.t the specification γ' and also time-stationary w.r.t M_τ . Then, for any translation-invariant measure ν' the entropic loss can be expressed via*

$$h(\nu'|\mu') - h(M_\tau\nu'|\mu') = \mathcal{H}_{M_\tau\nu'}(\hat{M}_{\tau,\nu'}|M_{-\tau}). \quad (11)$$

In [9] this result is proved for the case of a PCA with sitewise independent local updating. Here we extend it to our case of a weak PCA with quasilocal updating.

Proof: Let us suppress all primes in the notation. First notice, the error we make by replacing the starting configuration outside some finite volume is of boundary order. Indeed, let ξ , whenever it appears, be an arbitrary but fixed configuration in $\{0, \dots, q-1\}^G$, then

$$\frac{M_\tau(\sigma_\Lambda \xi_{\Lambda^c}, \eta_\Lambda)}{M_\tau(\sigma, \eta_\Lambda)} = \frac{\frac{\mu_{G \setminus \Lambda}[\xi_{G \setminus \Lambda}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\sigma_\Lambda} 1_{\eta_{\Lambda, \tau}}))}{\mu_{G \setminus \Lambda}[\xi_{G \setminus \Lambda}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\sigma_\Lambda}))}}{\frac{\mu_{G \setminus \Lambda}[\sigma_{G \setminus \Lambda}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\sigma_\Lambda} 1_{\eta_{\Lambda, \tau}}))}{\mu_{G \setminus \Lambda}[\sigma_{G \setminus \Lambda}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\sigma_\Lambda}))}} \leq e^{4 \sum_{A \cap \Lambda \neq \emptyset, A \cap \Lambda^c \neq \emptyset} \|\Phi_A\|} = e^{o(|\Lambda|)} \quad (12)$$

where of course we assumed $M_\tau(\sigma, \eta_\Lambda) \neq 0$.

Then since μ is assumed to be time-stationary the local entropic loss can be expressed in terms of the backwards transition operator and some error term of

boundary order. Indeed,

$$\begin{aligned}
& h_\Lambda(\nu|\mu) - h_\Lambda(M_\tau\nu|\mu) \\
&= \sum_{\eta_\Lambda \in \{1, \dots, q\}^\Lambda} M_\tau\nu(\eta_\Lambda) \log \frac{\mu(\eta_\Lambda)}{M_\tau\nu(\eta_\Lambda)} + \sum_{\sigma_\Lambda \in \{1, \dots, q\}^\Lambda} \nu(\sigma_\Lambda) \log \frac{\nu(\sigma_\Lambda)}{\mu(\sigma_\Lambda)} \\
&= \sum_{\eta_\Lambda} M_\tau\nu(\eta_\Lambda) \sum_{\sigma_\Lambda} Q_\nu(\sigma_\Lambda|\eta_\Lambda) \log \frac{M_\tau(\sigma_\Lambda \xi_{\Lambda^c}, \eta_\Lambda) \nu(\sigma_\Lambda)}{M_\tau\nu(\eta_\Lambda)} \frac{M_\tau\mu(\eta_\Lambda)}{M_\tau(\sigma_\Lambda \xi_{\Lambda^c}, \eta_\Lambda) \mu(\sigma_\Lambda)} \\
&= \sum_{\eta_\Lambda} M_\tau\nu(\eta_\Lambda) \sum_{\sigma_\Lambda} Q_\nu(\sigma_\Lambda|\eta_\Lambda) \log \frac{\int \nu(d\sigma) \frac{M_\tau(\sigma_\Lambda \xi_{\Lambda^c}, \eta_\Lambda)}{M_\tau(\sigma, \eta_\Lambda)} M_\tau(\sigma, \eta_\Lambda) 1_{\sigma_\Lambda}(\sigma)}{\int \mu(d\sigma) \frac{M_\tau(\sigma_\Lambda \xi_{\Lambda^c}, \eta_\Lambda)}{M_\tau(\sigma, \eta_\Lambda)} M_\tau(\sigma, \eta_\Lambda) 1_{\sigma_\Lambda}(\sigma)} \frac{M_\tau\mu(\eta_\Lambda)}{M_\tau\nu(\eta_\Lambda)} \\
&\leq \sum_{\eta_\Lambda} M_\tau\nu(\eta_\Lambda) \sum_{\sigma_\Lambda} Q_\nu(\sigma_\Lambda|\eta_\Lambda) \log \frac{Q_\nu(\sigma_\Lambda, \eta_\Lambda)}{M_\tau\nu(\eta_\Lambda)} \frac{M_\tau\mu(\eta_\Lambda)}{Q_\mu(\sigma_\Lambda, \eta_\Lambda)} + \log \frac{\sup_{\sigma_\Lambda = \tilde{\sigma}_\Lambda, \eta} \frac{M_\tau(\sigma, \eta_\Lambda)}{M_\tau(\tilde{\sigma}, \eta_\Lambda)}}{\inf_{\sigma_\Lambda = \tilde{\sigma}_\Lambda, \eta} \frac{M_\tau(\sigma, \eta_\Lambda)}{M_\tau(\tilde{\sigma}, \eta_\Lambda)}} \\
&= \sum_{\eta_\Lambda, \sigma_\Lambda} Q_\nu(\sigma_\Lambda, \eta_\Lambda) \log \frac{Q_\nu(\sigma_\Lambda|\eta_\Lambda)}{Q_\mu(\sigma_\Lambda|\eta_\Lambda)} + 2o(|\Lambda|)
\end{aligned}$$

where we used (12) and $\sum_{\eta_\Lambda \in \{1, \dots, q\}^\Lambda} M_\tau\nu(\eta_\Lambda) Q_\nu(\sigma_\Lambda|\eta_\Lambda) = \sum_{\eta_\Lambda} Q_\nu(\sigma_\Lambda, \eta_\Lambda) = \nu(\sigma_\Lambda)$. Notice, we get a similar bound from below, i.e

$$h_\Lambda(\nu|\mu) - h_\Lambda(M_\tau\nu|\mu) \geq \sum_{\eta_\Lambda, \sigma_\Lambda} Q_\nu(\sigma_\Lambda, \eta_\Lambda) \log \frac{Q_\nu(\sigma_\Lambda|\eta_\Lambda)}{Q_\mu(\sigma_\Lambda|\eta_\Lambda)} - o(|\Lambda|).$$

Together we have the following identity

$$\begin{aligned}
& h_\Lambda(\nu|\mu) - h_\Lambda(M_\tau\nu|\mu) = \mathbb{E}^{Q_\nu} \left[\log \frac{Q_\nu(\sigma_\Lambda|\eta_\Lambda)}{Q_\mu(\sigma_\Lambda|\eta_\Lambda)} \right] \pm o(|\Lambda|) \\
&= \mathbb{E}^{Q_\nu} \left[\log \frac{Q_\nu(\sigma_\Lambda|\eta_\Lambda)}{Q_\nu(\sigma_\Lambda|\eta)} \right] + \mathbb{E}^{Q_\nu} \left[\log \frac{\hat{M}_{\tau, \nu}(\eta, \sigma_\Lambda)}{M_{-\tau}(\eta, \sigma_\Lambda)} \right] + \mathbb{E}^{Q_\nu} \left[\log \frac{Q_\mu(\sigma_\Lambda|\eta)}{Q_\mu(\sigma_\Lambda|\eta_\Lambda)} \right] \pm o(|\Lambda|).
\end{aligned} \tag{13}$$

Under the volume limit, the l.h.s of (13) becomes the l.h.s of (11) and for the second summand on the r.h.s of (13) we have

$$\begin{aligned}
\limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \mathbb{E}^{Q_\nu} \left[\log \frac{\hat{M}_{\tau, \nu}(\eta, \sigma_\Lambda)}{M_{-\tau}(\eta, \sigma_\Lambda)} \right] &= \limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \int M_\tau\nu(d\eta) h_\Lambda[\hat{M}_{\tau, \nu}(\eta, \cdot) | M_{-\tau}(\eta, \cdot)] \\
&= \mathcal{H}_{M_\tau\nu}(\hat{M}_{\tau, \nu} | M_{-\tau})
\end{aligned}$$

which is the r.h.s of (11). Hence, in order to prove (11) it suffices to show, that the first and the third summand on the r.h.s of (13) are $o(|\Lambda|)$ -functions. Since the third summand is not a special case of the first summand, we have to proceed separately.

For the first summand in (13) we can follow closely the arguments from [9] Proposition 2.1. For the readers convenience we provide them here as well.

Let $\{i_1, \dots, i_{|\Lambda|}\}$ be the lexicographic ordering of the elements of Λ and define $\Lambda_k = \{i_1, \dots, i_k\}$ for $1 \leq k \leq \Lambda$ with $\Lambda_0 = \emptyset$. By Bayes' theorem we have

$$Q_\nu(\sigma_\Lambda | \eta_\Lambda) = Q_\nu(\sigma_{i_1} | \eta_\Lambda) Q_\nu(\sigma_{i_2} | \eta_\Lambda, \sigma_{i_1}) \cdots Q_\nu(\sigma_{i_{|\Lambda|}} | \eta_\Lambda, \sigma_{\Lambda_{|\Lambda|-1}})$$

and hence we can write

$$\log Q_\nu(\sigma_\Lambda | \eta_\Lambda) = \sum_{k=1}^{|\Lambda|} \log Q_\nu(\sigma_{i_k} | \eta_\Lambda, \sigma_{\Lambda_{k-1}}).$$

By translation invariance of Q_ν we have

$$\mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_{i_k} | \eta_\Lambda, \sigma_{\Lambda_{k-1}})] = \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_{\theta_{-i_k}\Lambda}, \sigma_{\theta_{-i_k}\Lambda_{k-1}})]$$

where $\theta_i\Lambda$ denotes a lattice translation of the set $\Lambda \subset G$ by $i \in G$. We want to use the Shannon-Breiman-McMillan Theorem from [4] as used in [9] and therefore understand the conditional densities as a sequence of densities belonging to a stochastic process under the invariant measure Q_ν .

Let $G_- := \{i \in G : i \prec 0\}$, where \prec is the lexicographic order. By the Shannon-Breiman-McMillan Theorem the stationary and ergodic process has the "Asymptotic Equipartition Property" and thus the sequence of conditional measures has an almost sure limit. In particular it is a Cauchy sequence and hence, for every $\varepsilon > 0$ there are $A \subset G, B \subset G_-$ finite such that if $A \subset V$ and $B \subset W \subset G_-$ we have

$$\left| \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_V, \sigma_W)] - \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_A, \sigma_B)] \right| < \varepsilon.$$

Also notice for all Λ and i_k there is the simple bound

$$\mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_{\theta_{-i_k}\Lambda}, \sigma_{\theta_{-i_k}\Lambda_{k-1}})] \leq \log q.$$

Now we can separate bulk and boundary terms. This gives

$$\begin{aligned} & \limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \sum_{k=1}^{|\Lambda|} \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_{\theta_{-i_k}\Lambda}, \sigma_{\theta_{-i_k}\Lambda_{k-1}})] \\ &= \limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \left\{ \sum_{k: A \subset \theta_{-i_k}\Lambda, B \subset \theta_{-i_k}\Lambda_{k-1}} \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_{\theta_{-i_k}\Lambda}, \sigma_{\theta_{-i_k}\Lambda_{k-1}})] \right. \\ & \quad \left. + |\{k : A \not\subset \theta_{-i_k}\Lambda, B \not\subset \theta_{-i_k}\Lambda_{k-1}\}| \log q \right\}. \end{aligned} \tag{14}$$

For large Λ the first term on the r.h.s of (14) contains the bulk of the summands. The second term on the r.h.s of (14) is of boundary order. Hence

$$\begin{aligned} \limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_\Lambda | \eta_\Lambda)] &\leq \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_A, \sigma_B)] + \varepsilon \quad \text{and} \\ \limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_\Lambda | \eta_\Lambda)] &\geq \mathbb{E}^{Q_\nu}[\log Q_\nu(\sigma_0 | \eta_A, \sigma_B)] - \varepsilon. \end{aligned}$$

Letting ε go to zero we have

$$\limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \mathbb{E}^{Q_\nu} [\log Q_\nu(\sigma_\Lambda | \eta_\Lambda)] = \mathbb{E}^{Q_\nu} [\log Q_\nu(\sigma_0 | \eta, \sigma_{G_-})].$$

Using the same arguments as above one also shows

$$\limsup_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \mathbb{E}^{Q_\nu} [\log Q_\nu(\sigma_\Lambda | \eta)] = \mathbb{E}^{Q_\nu} [\log Q_\nu(\sigma_0 | \eta, \sigma_{G_-})].$$

For the second summand in (13) recall equation (12). Of course also $M_{-\tau}$ has such an error bound and we have

$$\frac{M_{-\tau}(\eta_\Lambda \xi_{\Lambda^c}, \sigma_\Lambda)}{M_{-\tau}(\eta, \sigma_\Lambda)} \leq e^{o(|\Lambda|)}.$$

Hence with Lemma 3.1 we can write

$$\frac{Q_\mu(\sigma_\Lambda | \eta_\Lambda)}{Q_\mu(\sigma_\Lambda | \eta)} = \int M_\tau \mu(d\tilde{\eta} | \eta_\Lambda) \frac{M_{-\tau}(\eta_\Lambda \tilde{\eta}_{\Lambda^c}, \sigma_\Lambda)}{M_{-\tau}(\eta, \sigma_\Lambda)} \leq e^{o(|\Lambda|)}$$

and thus $\mathbb{E}^{Q_\nu} [\log \frac{Q_\mu(\sigma_\Lambda | \eta)}{Q_\mu(\sigma_\Lambda | \eta_\Lambda)}] = \pm o(|\Lambda|)$. As a result, in the infinite-volume limit, the $o(|\Lambda|)$ terms vanish and this completes the proof. \square

Recall that our goal is to show that time-stationary measures must be Gibbs measures. The following result is slightly more general. Using the characterization of the entropic loss from the preceding lemma we will show that zero entropic loss implies the single site DLR equation and thus the Gibbs property. The main tools for the proof are an adaptation of the Gibbs variational principle and an argument from [40] to infer the DLR equation from the backwards transition operator. For more general background about the Gibbs variational principle also in the context of generalized Gibbsian measures see [27, 37, 19].

Theorem 3.3 *Suppose that μ' is a translation-invariant Gibbs measure w.r.t the specification γ' and also time-stationary w.r.t M_τ where $0 < \tau < 2\pi/q$. Then, for any translation-invariant measure ν' with*

$$h(\nu' | \mu') = h(M_\tau \nu' | \mu')$$

we have that ν' is also a Gibbs measure for the same specification, i.e $\nu' \in \mathcal{G}_\theta(\gamma')$. Conversely if $\nu' \in \mathcal{G}_\theta(\gamma')$ then also $h(\nu' | \mu') = h(M_\tau \nu' | \mu')$.

Please note, Theorem 3.3 and the following results are valid for general $\tau \in \mathbb{R}$. We set $0 < \tau < 2\pi/q$ only to keep the presentation more transparent.

Proof: Let us again suppress all primes in the notation whenever unambiguous.

Step 1: We show that

$$\hat{M}_{\tau,\nu}(\eta, \sigma_\Lambda) = M_{-\tau}(\eta, \sigma_\Lambda) \quad \text{for } M_{\tau,\nu}\text{-a.a. } \eta \quad (15)$$

under the hypothesis of the theorem. To do so we extend the proof of the Gibbs variational principle in [27] Chapter 15.4. to our situation where we have to deal with an additional dependence on the future configuration η distributed according to $M_{\tau,\nu}$. Define $h_\Lambda^\eta(\hat{M}_\nu|\hat{M}_\mu) := h_\Lambda[\hat{M}_{\tau,\nu}(\eta, \cdot)|M_{-\tau}(\eta, \cdot)]$, then by Fatou's Lemma and Lemma 3.2 we have

$$\begin{aligned} 0 = \mathcal{H}_{M_{\tau,\nu}}(\hat{M}_\nu|\hat{M}_\mu) &= \int \nu(d\eta) \limsup_{\Lambda} \frac{1}{|\Lambda|} h_\Lambda^\eta(\hat{M}_\nu|\hat{M}_\mu) \\ &\geq \limsup_{\Lambda} \frac{1}{|\Lambda|} \int M_{\tau,\nu}(d\eta) h_\Lambda^\eta(\hat{M}_\nu|\hat{M}_\mu) = 0. \end{aligned} \quad (16)$$

In particular for all cofinal sequences of finite sets $0 = \lim_{\Lambda_n \uparrow G} \frac{1}{|\Lambda_n|} \int M_{\tau,\nu}(d\eta) h_{\Lambda_n}^\eta(\hat{M}_\nu|\hat{M}_\mu)$. Since the local relative entropy $h_\Lambda^\eta(\hat{M}_\nu|\hat{M}_\mu)$ must be finite for all Λ and $M_{\tau,\nu}$ -a.a. η there exists a density

$$f_\Lambda^\eta := \frac{d\hat{M}_\nu(\eta, \cdot|_\Lambda)}{d\hat{M}_\mu(\eta, \cdot|_\Lambda)}$$

depending on configurations inside Λ only.

Step 1a: We show that expectations of local entropy densities behave nice w.r.t volumes in the following sense: For any $\delta > 0$ and any cube $C \supset \Lambda$ there exists a finite set Δ with $\Delta \supset C$ such that

$$\int M_{\tau,\nu}(d\eta) h_\Delta^\eta(\hat{M}_\nu|\hat{M}_\mu) - \int M_{\tau,\nu}(d\eta) h_{\Delta \setminus \Lambda}^\eta(\hat{M}_\nu|\hat{M}_\mu) \leq \delta.$$

The proof is an integrated version of the first step of the proof of the variational principle in [27] Chapter 15.4. Indeed, by (16) we can find $n \geq 1$ such that for the centered hypercube Λ_n we have $|\Lambda_n| \geq |C|$ and $|\Lambda_n|^{-1} \int M_{\tau,\nu}(d\eta) h_{\Lambda_n}^\eta(\hat{M}_\nu|\hat{M}_\mu) \leq \delta/2^d|C|$. Now we can choose an integer $m \geq 1$ in such a way that

$$m^d|C| \leq |\Lambda_n| \leq (2m)^d|C|.$$

Further let us choose m^d lattice sites $i(1), \dots, i(m^d)$ in such a way that the translates $C(k) = C + i(k)$, $1 \leq k \leq m^d$ are pairwise disjoint subsets of Λ_n . For each $1 \leq k \leq m^d$ we put $W(k) = C(1) \cup \dots \cup C(k)$ and $\Lambda(k) = \Lambda + i(k)$. Then using the monotonicity of the relative density we can write

$$\begin{aligned} &\frac{1}{m^d} \sum_{i=1}^{m^d} \left[\int M_{\tau,\nu}(d\eta) h_{W(k)}^\eta(\hat{M}_\nu|\hat{M}_\mu) - \int M_{\tau,\nu}(d\eta) h_{W(k) \setminus \Lambda(k)}^\eta(\hat{M}_\nu|\hat{M}_\mu) \right] \\ &\leq \frac{1}{m^d} \sum_{i=1}^{m^d} \left[\int M_{\tau,\nu}(d\eta) h_{W(k)}^\eta(\hat{M}_\nu|\hat{M}_\mu) - \int M_{\tau,\nu}(d\eta) h_{W(k) \setminus C(k)}^\eta(\hat{M}_\nu|\hat{M}_\mu) \right] \\ &= \frac{1}{m^d} \int M_{\tau,\nu}(d\eta) h_{W(m^d)}^\eta(\hat{M}_\nu|\hat{M}_\mu) \\ &\leq \frac{1}{m^d} \int M_{\tau,\nu}(d\eta) h_{\Lambda_n}^\eta(\hat{M}_\nu|\hat{M}_\mu) \leq 2^d|C| |\Lambda_n|^{-1} \int M_{\tau,\nu}(d\eta) h_{\Lambda_n}^\eta(\hat{M}_\nu|\hat{M}_\mu) \leq \delta. \end{aligned}$$

Consequently, there exists an index k such that

$$\int M_\tau \nu(d\eta) h_{W(k)}^\eta(\hat{M}_\nu | \hat{M}_\mu) - \int M_\tau \nu(d\eta) h_{W(k) \setminus \Lambda(k)}^\eta(\hat{M}_\nu | \hat{M}_\mu) \leq \delta.$$

The claim of Step 1a thus follows by putting $\Delta := W(k) - i(k)$ and using the translation-invariance of $\hat{M}_\nu(\eta, \cdot |_\Lambda)$ and $\hat{M}_\mu(\eta, \cdot |_\Lambda)$ under $M_\tau \nu(d\eta)$.

Step 1b: We want to show that closeness of integrated entropy densities implies closeness of integrated densities. This again is an integrated version of the analogous statement in [27] Chapter 15.4. More precisely we show that for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\int M_\tau \nu(d\eta) \int \hat{M}_\mu(d\sigma, \eta) |f_\Delta^\eta(\sigma) - f_{\Delta \setminus \Lambda}^\eta(\sigma)| \leq \varepsilon$$

whenever $\Lambda \subset \Delta$ and

$$\int M_\tau \nu(d\eta) h_\Delta^\eta(\hat{M}_\nu | \hat{M}_\mu) - \int M_\tau \nu(d\eta) h_{\Delta \setminus \Lambda}^\eta(\hat{M}_\nu | \hat{M}_\mu) \leq \delta.$$

Indeed, let $\psi(x) := x \log x - x + 1$ for $x \geq 0$, then

$$\begin{aligned} & \int M_\tau \nu(d\eta) h_\Delta^\eta(\hat{M}_\nu | \hat{M}_\mu) - \int M_\tau \nu(d\eta) h_{\Delta \setminus \Lambda}^\eta(\hat{M}_\nu | \hat{M}_\mu) \\ &= \int M_\tau \nu(d\eta) \int \hat{M}_\nu(\eta, d\sigma) \log \frac{f_\Delta^\eta(\sigma)}{f_{\Delta \setminus \Lambda}^\eta(\sigma)} \\ &= \int M_\tau \nu(d\eta) \int \hat{M}_\mu(\eta, d\sigma) f_{\Delta \setminus \Lambda}^\eta(\sigma) \psi\left(\frac{f_\Delta^\eta(\sigma)}{f_{\Delta \setminus \Lambda}^\eta(\sigma)}\right). \end{aligned}$$

Notice, there is a number $0 < r < \infty$ such that $|x - 1| \leq r\psi(x) + \varepsilon/2$ for all $x \geq 0$. By putting $\delta = \varepsilon/2r$ we can thus write

$$\begin{aligned} & \int M_\tau \nu(d\eta) \int \hat{M}_\mu(d\sigma, \eta) |f_\Delta^\eta(\sigma) - f_{\Delta \setminus \Lambda}^\eta(\sigma)| \\ &= \int M_\tau \nu(d\eta) \int \hat{M}_\mu(d\sigma, \eta) f_{\Delta \setminus \Lambda}^\eta(\sigma) \left| \frac{f_\Delta^\eta(\sigma)}{f_{\Delta \setminus \Lambda}^\eta(\sigma)} - 1 \right| \\ &\leq r \int M_\tau \nu(d\eta) \int \hat{M}_\mu(\eta, d\sigma) f_{\Delta \setminus \Lambda}^\eta(\sigma) \psi\left(\frac{f_\Delta^\eta(\sigma)}{f_{\Delta \setminus \Lambda}^\eta(\sigma)}\right) + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

Step 1c: To prove the desired result for the Step 1, i.e that $M_\tau \nu$ -a.s. we have $\hat{M}_{\tau, \nu}(\eta, \sigma_\Lambda) = M_{-\tau}(\eta, \sigma_\Lambda)$, we show the DLR equation for the backwards operator corresponding to ν for the specification given by the backwards transition operator corresponding to the Gibbs measure μ . Indeed, let g be a local test-function and $\varepsilon > 0$. Since μ is a Gibbs measure there exists a local function \tilde{g}^η with support independent of η (which depend only on configurations outside Λ) such that $\sup_\eta \|\gamma_\Lambda^\eta(g|\cdot) - \tilde{g}^\eta(\cdot)\| < \varepsilon$ where

$$\gamma_\Lambda^\eta(d\tilde{\sigma} | \sigma_{\Lambda^c}) := Q_\mu(d\tilde{\sigma} | \sigma_{\Lambda^c}, \eta).$$

To see this we note that $M_{-\tau}(\eta, d\sigma)$ as a measure on the σ is in the Dobrushin region uniformly in η . This is true by arguments provided in the proof of Step 2 below equation (19). Let $C \supset \Lambda$ be a cube such that g depends only on configuration on C and \tilde{g}^η depends only on $C \setminus \Lambda$. Choose δ in terms of ε as in Step 1b, and define Δ in terms of C and δ as in Step 1a.

$$\begin{aligned}
& \int M_{\tau}\nu(d\eta) \left| \int \hat{M}_{\tau,\nu}(\eta, d\sigma) g(\sigma) - \int \hat{M}_{\tau,\nu}(\eta, d\sigma) \gamma_{\Lambda}^{\eta}(g|\sigma) \right| \\
& \leq \int M_{\tau}\nu(d\eta) \left[\int \hat{M}_{\tau,\nu}(\eta, d\sigma) |\gamma_{\Lambda}^{\eta}(g|\sigma) - \tilde{g}^{\eta}(\sigma)| \right. \\
& \quad + \left| \int \hat{M}_{\tau,\nu}(\eta, d\sigma) \tilde{g}^{\eta}(\sigma) - \int M_{-\tau}(\eta, d\sigma) f_{\Delta \setminus \Lambda}^{\eta}(\sigma) \tilde{g}^{\eta}(\sigma) \right| \\
& \quad + \int M_{-\tau}(\eta, d\sigma) f_{\Delta \setminus \Lambda}^{\eta}(\sigma) |\tilde{g}^{\eta}(\sigma) - \gamma_{\Lambda}^{\eta}(g|\sigma)| \\
& \quad + \left| \int M_{-\tau}(\eta, d\sigma) (f_{\Delta \setminus \Lambda}^{\eta}(\sigma) (\gamma_{\Lambda}^{\eta}(g|\sigma) - g(\sigma))) \right| \\
& \quad + \|g\| \int M_{-\tau}(\eta, d\sigma) |f_{\Delta \setminus \Lambda}^{\eta}(\sigma) - f_{\Delta}^{\eta}(\sigma)| \\
& \quad \left. + \left| \int M_{-\tau}(\eta, d\sigma) f_{\Delta}^{\eta}(\sigma) g(\sigma) - \int \hat{M}_{\tau,\nu}(\eta, d\sigma) g(\sigma) \right| \right]
\end{aligned}$$

Since \tilde{g}^{η} depends only on $\Delta \setminus \Lambda$ and g depends only Δ , the second and the last term on the right are zero. The fourth term vanishes because $\int M_{-\tau}(\eta, d\sigma) \int \gamma^{\eta}(d\tilde{\sigma}|\sigma) = \int M_{-\tau}(\eta, d\sigma)$ and $f_{\Delta \setminus \Lambda}^{\eta}$ depends only on Λ^c . Due to the choice of g , the first and the third term are each at most ε . The only non-trivial term is the fifth one. This term is not larger than $\|g\|\varepsilon$ because of our choice of Δ see Step 1b. As ε was arbitrary, we conclude that $\int M_{\tau}\nu(d\eta) \left| \int \hat{M}_{\tau,\nu}(\eta, d\sigma) g(\sigma) - \int \hat{M}_{\tau,\nu}(\eta, d\sigma) \gamma_{\Lambda}^{\eta}(g|\sigma) \right| = 0$. Hence $M_{\tau}\nu$ -a.s. we have $\int \hat{M}_{\tau,\nu}(d\sigma, \eta) g(\sigma) = \int \hat{M}_{\tau,\nu}(d\sigma, \eta) \gamma_{\Lambda}^{\eta}(g|\sigma)$. And thus $\hat{M}_{\tau,\nu}$ is the unique Gibbs measure for γ_{Λ}^{η} . In particular equation (15) follows.

Step 2: As a consequence of Step 1, equation (15) holds and we can conclude that Q_{ν} -a.s.

$$Q_{\nu}(\sigma_{\Lambda}|\sigma_{\Lambda^c}, \eta) = Q_{\mu}(\sigma_{\Lambda}|\sigma_{\Lambda^c}, \eta).$$

In what follows, we use the basic idea from [40] to infer from this the single-site DLR equation for ν . Notice, for $i \in G$, finite sets $\Lambda \subset G \setminus i$, $\bar{\Lambda} \subset G$ and $\tilde{\sigma}_{i^c} = \sigma_{i^c}$ we have

$$\mu(\sigma_i|\sigma_{\Lambda}) Q_{\mu}(\eta_{\bar{\Lambda}}|\sigma_{\Lambda \cup i}) Q_{\mu}(\tilde{\sigma}_i|\sigma_{\Lambda}, \eta_{\bar{\Lambda}}) = \mu(\tilde{\sigma}_i|\sigma_{\Lambda}) Q_{\mu}(\eta_{\bar{\Lambda}}|\tilde{\sigma}_{\Lambda \cup i}) Q_{\mu}(\sigma_i|\sigma_{\Lambda}, \eta_{\bar{\Lambda}})$$

where μ is a Gibbs measure for the specification γ' given in (5). Letting Λ go to $G \setminus i$ and $\bar{\Lambda}$ go to G we get Q_{μ} -a.s.

$$\frac{\mu(\sigma_i|\sigma_{i^c})}{\mu(\tilde{\sigma}_i|\sigma_{i^c})} = \frac{\gamma'(\sigma_i|\sigma_{i^c})}{\gamma'(\tilde{\sigma}_i|\sigma_{i^c})} = \lim_{\bar{\Lambda} \uparrow G} \frac{M_{\tau}(\tilde{\sigma}, \eta_{\bar{\Lambda}})}{M_{\tau}(\sigma, \eta_{\bar{\Lambda}})} \frac{Q_{\mu}(\sigma_i|\sigma_{i^c}, \eta)}{Q_{\mu}(\tilde{\sigma}_i|\sigma_{i^c}, \eta)} \quad (17)$$

where we note that Q_μ -a.s. η is accessible for $\sigma, \tilde{\sigma}$ and vice versa. Also we will prove right below that the $\lim_{\bar{\Lambda} \uparrow G} \frac{M_\tau(\tilde{\sigma}, \eta_{\bar{\Lambda}})}{M_\tau(\sigma, \eta_{\bar{\Lambda}})}$ exists. In the same way one gets Q_ν -a.s.

$$\frac{\nu(\sigma_i | \sigma_{i^c})}{\nu(\tilde{\sigma}_i | \sigma_{i^c})} = \lim_{\bar{\Lambda} \uparrow G} \frac{M_\tau(\tilde{\sigma}, \eta_{\bar{\Lambda}})}{M_\tau(\sigma, \eta_{\bar{\Lambda}})} \frac{Q_\nu(\sigma_i | \sigma_{i^c}, \eta)}{Q_\nu(\tilde{\sigma}_i | \sigma_{i^c}, \eta)} \quad (18)$$

and hence Q_ν -a.s. $\frac{\nu(\sigma_i | \sigma_{i^c})}{\nu(\tilde{\sigma}_i | \sigma_{i^c})} = \frac{\gamma'(\sigma_i | \sigma_{i^c})}{\gamma'(\tilde{\sigma}_i | \sigma_{i^c})}$. Summing over the $\tilde{\sigma}_i$ we get the desired single-site DLR-equation

$$\nu(\sigma_i | \sigma_{i^c}) = \gamma'(\sigma_i | \sigma_{i^c})$$

Q_ν -a.s. and thus ν is a Gibbs measure for γ' .

What remains to be proved is that the limit in (17) and (18) indeed exists. Let us therefore consider M_τ , $\tilde{\sigma}'_{i^c} = \sigma'_{i^c}$ and η'_Λ such that $M_\tau(\sigma', \eta'_\Lambda) \neq 0 \neq M_\tau(\tilde{\sigma}', \eta'_\Lambda)$ for all $\bar{\Lambda}$. Then we have, using the notation from (6)

$$\begin{aligned} \frac{M_\tau(\tilde{\sigma}', \eta'_\Lambda)}{M_\tau(\sigma', \eta'_\Lambda)} &= \frac{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\tilde{\sigma}'_i} 1_{\eta'_{\Lambda, \tau}}))}{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\sigma'_i} 1_{\eta'_{\Lambda, \tau}}))} \frac{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\sigma'_i}))}{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\tilde{\sigma}'_i}))} \\ &= \frac{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\tilde{\sigma}'_i} 1_{\eta'_{\Lambda, \tau}}) | 1_{\eta'_{\Lambda \setminus i, \tau}})}{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\sigma'_i} 1_{\eta'_{\Lambda, \tau}}) | 1_{\eta'_{\Lambda \setminus i, \tau}})} \frac{\gamma'(\sigma'_i | \sigma'_{G \setminus i})}{\gamma'(\tilde{\sigma}'_i | \sigma'_{G \setminus i})}. \end{aligned}$$

Now since we are in the Dobrushin region already for the coarse-graining T with $q \geq q_0(\Phi)$, to coarse-grain with

$$[0, 2\pi) \mapsto \{[0, \tau), [\tau, 2\pi/q), [2\pi/q, 2\pi/q + \tau), \dots, [2\pi - \tau, 2\pi)\} \quad (19)$$

is even finer and thus the conditional first-layer specifications are even more in the Dobrushin region. The idea here is that by conditioning to configurations of small segments of the sphere, one can control the possible interaction strength between spins in such a way, that the conditional specification in the first-layer model is in a parameter regime where there is a unique conditional Gibbs measure. Since the individual segments of the sphere under equal-arc discretization with q become even smaller when we discretize the sphere as in (19) a fortiori the Dobrushin condition of weak interaction is satisfied. In particular there is a unique limiting measure on the conditioning $\sigma'_{G \setminus i} \cap \eta'_{G \setminus i, \tau}$. In other words

$$\begin{aligned} &\lim_{\bar{\Lambda} \uparrow G} \frac{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\tilde{\sigma}'_i} 1_{\eta'_{\Lambda, \tau}}) | 1_{\eta'_{\Lambda \setminus i, \tau}}))}{\mu_{G \setminus i}[\sigma'_{G \setminus i}](\lambda(e^{-H_i} 1_{\sigma'_i} 1_{\eta'_{\Lambda, \tau}}) | 1_{\eta'_{\Lambda \setminus i, \tau}}))} \\ &= \frac{\mu_{G \setminus i}[\sigma'_{G \setminus i} \cap \eta'_{G \setminus i, \tau}](\lambda(e^{-H_i} 1_{\tilde{\sigma}'_i} 1_{\eta'_{\Lambda, \tau}}))}{\mu_{G \setminus i}[\sigma'_{G \setminus i} \cap \eta'_{G \setminus i, \tau}](\lambda(e^{-H_i} 1_{\sigma'_i} 1_{\eta'_{\Lambda, \tau}}))} \end{aligned}$$

exists where the limiting measure $\mu_{G \setminus i}[\sigma'_{G \setminus i} \cap \eta'_{G \setminus i, \tau}]$ is the unique Gibbs measure conditional to the configuration $\sigma'_{G \setminus i} \cap \eta'_{G \setminus i, \tau}$ which is in the τ -dependent coarse-graining (19).

Step 3: The converse statement follows from the fact that if $\nu' \in \mathcal{G}_\theta(\gamma')$ then $\hat{M}_{\tau, \nu'} = M_{-\tau}$ by Lemma 3.1 and hence the l.h.s of equation (11) in Lemma 3.2 is zero. We conclude $h(\nu'|\mu') = h(M_\tau \nu'|\mu')$. \square

Notice, the uniform mixture μ'_* as in (9) is a Gibbs measure for γ' , translation-invariant and time-stationary. Hence we can apply Theorem 3.3 to our model with $\mu' = \mu'_*$ and the following corollary holds.

Corollary 3.4 *Take $0 < \tau < \frac{2\pi}{q}$ fixed, then the lattice-translation invariant measures which are invariant under the one time-step updating with the transition operator M_τ must be contained in the lattice-translation invariant measures $\mathcal{G}_\theta(\gamma')$.*

Let us come to the main conclusion regarding ergodicity and uniqueness of time-stationary measures for our model.

4 Stationary measures and rotating states

Notice that we have assembled essentially two properties of our process. In simple words those are: Translation-invariant time-stationary measures must be Gibbs measures and translation-invariant Gibbs measures rotate. The rotation property alone already induces non-ergodicity. Using the combination of the two properties we can draw conclusions about the number of time-stationary measures and prove part 3 in Theorem 1.1.

Theorem 4.1 *Let $0 < \tau < \frac{2\pi}{q}$ then there are two scenarios.*

1. *If $\frac{\tau}{2\pi}$ is rational, the Markov process with transition operator M_τ has a continuum of translation-invariant and time-stationary measures. Further the dynamics is non-ergodic with periodic orbits given by*

$$\Omega_\alpha := \{\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma') : \varphi = n\tau + \alpha \text{ for some } n \in \mathbb{N}\}.$$

2. *If $\frac{\tau}{2\pi}$ is irrational, the process has a unique translation-invariant and time-stationary measure $\mu'_* = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \mu'_\varphi$ where $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$. The dynamics is non-ergodic since $M_\tau^n \mu'_\varphi \not\rightarrow \mu'_*$ as $n \rightarrow \infty$. All orbits are dense in $\text{ex } \mathcal{G}_\theta(\gamma')$ w.r.t the weak topology.*

Proof: By Corollary 3.4 the investigation of translation-invariant and time-stationary measures can be restricted to the set $\mathcal{G}_\theta(\gamma')$.

In case $\frac{\tau}{2\pi}$ is rational there exist integers n, m such that $n\tau = m2\pi$. Hence for all $0 \leq \alpha < 2\pi$ the equal weight measure $\frac{1}{n} \sum_{k=1}^n \mu'_{k\tau+\alpha}$ is time-stationary where $\mu'_{k\tau+\alpha}$ denotes the extremal translation-invariant Gibbs measure with order parameter $(k\tau + \alpha) \bmod_{2\pi} \in S^1$. Further we have $M_\tau^n \mu'_\varphi = \mu'_{\varphi+n\tau} = \mu'_{\varphi+m2\pi} = \mu'_\varphi$ for all $\mu'_\varphi \in \text{ex } \mathcal{G}_\theta(\gamma')$ and hence for a given starting measure μ'_φ the set $\{\mu'_\psi \in \text{ex } \mathcal{G}_\theta(\gamma') : \psi = n\tau\varphi \text{ for some } n \in \mathbb{N}\}$ is a periodic orbit for the dynamics. In particular the process can not be ergodic.

If $\frac{\tau}{2\pi}$ is irrational the only possible symmetrically mixed measure as in the first case is the measure μ'_* . A more detailed measure-theoretic proof for this fact can be found in [31] Proposition 5.1 where one can replace the Markov semigroup by $(M_\tau^n)_{n \in \mathbb{N}}$. For the second statement notice, $M_\tau^n \mu'_\varphi = \mu'_{\varphi+n\tau} \in \text{ex } \mathcal{G}_\theta(\gamma')$ for all $n \in \mathbb{N}$, but $\mu'_* \notin \text{ex } \mathcal{G}_\theta(\gamma')$ and hence $M_\tau^n \mu'_\varphi \not\rightarrow \mu'_*$. For the third statement realize that for any $0 \leq \varphi, \psi < 2\pi$ there exists a subsequence such that $(\psi + n_k \tau) \bmod_{2\pi} \rightarrow \varphi$ for $k \rightarrow \infty$. Hence it suffices to show $\mu'_\varphi \rightarrow \mu'_0$ weakly for $\varphi \rightarrow 0$. But this is true since for the test-functions $1_{\omega'_\Lambda}$ we have

$$\lim_{\varphi \rightarrow 0} \mu'_\varphi(\omega'_\Lambda) = \lim_{\varphi \rightarrow 0} \int \mu_\varphi(d\omega) 1_{\omega'_\Lambda}(\omega) = \int \mu_0(d\omega) \lim_{\varphi \rightarrow 0} \gamma_\Lambda(1_{\omega'_\Lambda - \varphi} | \omega_{\Lambda^c})$$

where we used the DLR-equation and the dominated convergence theorem. Now by the continuity of the Hamiltonian and the fact, that the a priori measures is the Lebesgue measure on S^1 we have $\lim_{\varphi \rightarrow 0} \gamma_\Lambda(1_{\omega'_\Lambda - \varphi} | \omega_{\Lambda^c}) = \gamma_\Lambda(1_{\omega'_\Lambda} | \omega_{\Lambda^c})$ for any boundary condition ω_{Λ^c} . In particular the process can again not be ergodic. \square

Proof of Theorem 1.1: The proofs of the locality properties in the past and in the future are given in Propositions 2.3 and 2.4. For $0 \leq \tau \leq 2\pi/q$ the updating is Bernoulli as can be seen from the representation (7). Further defining $\mathcal{I} := \text{ex } \mathcal{G}_\theta(\gamma')$ in Corollary 2.2 we prove the desired properties of the function ψ (this is part 1). The rotation property is proved to hold in Proposition 2.5 (this is part 2). The two possible scenarios for time-stationary measures and periodic orbits are proved in Theorem 4.1 (this is part 3). \square

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